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CLASSIFICATION OF LINEAR INTEGRALS OF A HOLONOMIC MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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Il. ILIEV
(Plovdiv)
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The linear integrals of a mechanical system are classified according to the solutions of the Killing equation and of the form of the generalized forces. An example of a mechanical system with two degrees of freedom which has a generalized force function but no linear integral, was given in [1].

Let $\lambda_\varkappa \dot{q}^\varkappa = c$ be the linear integral of a mechanical system with two degrees of freedom. This requires that [1]

$$\nabla_s \lambda_\varkappa + \nabla_\varkappa \lambda_s = 0 \tag{1}$$

$$\lambda_\varkappa Q^\varkappa = 0 \tag{2}$$

Considering

$$2T dt^2 = ds^2 = g_{\lambda\mu} dq^\lambda dq^\mu$$

as a linear element of the two-dimensional Riemannian space V_2 we find, that the following possibilities [2, 3] may be given to the Killing equation (1). Equations (1) have:

a) no solution,

b) one solution, or

c) three solutions.

In the case (a) Eq. (1) has no solution, hence the mechanical system has no linear integral. In the cases (b) and (c), using integrable transformations the linear element can be reduced to the form

$$2T\, dt^2 = ds^2 = V(q^1)\,[(dq^1)^2 + (dq^2)^2] \tag{3}$$

then it is said that the rotation metric is given [2]. It has been shown that each rotation metric defines a surface of revolution.

In the case (b), $\lambda_1 = 0, \qquad \lambda_2 = V$

is a unique solution of (1) defining a vector parallel to the surface. Condition (1) implies that $Q^2 = 0$, i.e. that the generalized force acts along the meridian, is the necessary condition for a linear system to have a linear integral. When this condition holds, the system has a single linear integral, otherwise it has no such integral.

In the case (c) the Riemannian space has a constant Gaussian curvature K, and Eqs. (1) have three independent solutions. A method for obtaining these solutions for V_2 with a metric of the form (3) was given in [3]. Let us denote these solutions by $\lambda_\varkappa{}^i; i = 1, 2, 3$, where $\varkappa = 1, 2$. Any solution $\lambda_\varkappa$ can be written as a linear combination of $\lambda_\varkappa{}^i$, namely

$$\lambda_\varkappa = \lambda_\varkappa{}^1 T_1 + \lambda_\varkappa{}^2 T_2 + \lambda_\varkappa{}^3 T_3, \qquad T_1, T_2, T_3 = \text{const}$$

When $Q^1 = Q^2 = 0$, the mechanical system has three independent linear integrals

$$\lambda_\varkappa{}^1 q^{\cdot\varkappa} = c^1, \qquad \lambda_\varkappa{}^2 q^{\cdot\varkappa} = c^2, \qquad \lambda_\varkappa{}^3 q^{\cdot\varkappa} = c^3$$

If at least one $Q^\varkappa$ is not zero, then

$$Q^1 = -\rho\,(T_1\lambda_2{}^1 + T_2\lambda_2{}^2 + T_3\lambda_2{}^3), \qquad Q^2 = \rho\,(T_1\lambda_1{}^1 + T_2\lambda_1{}^2 + T_3\lambda_1{}^3) \qquad (4)$$

is the necessary condition for the mechanical system to have a linear integral, and in this case it has a single linear integral. Suppose that another independent solution $\mu_\varkappa$ of (1) exists satisfying (4). This solution will have to be collinear with $\lambda_\varkappa$. But this was shown in [1] to be impossible. When condition (4) does not hold, the mechanical system has no linear integral.

Let us consider the case (b) in more detail. The following theorem shows what form the force function U must have for the mechanical system to have a linear integral.

Theorem. The necessary and sufficient condition for a mechanical system in the case (b) to have a linear integral, is, that the force function U is a function of the Gaussian curvature K, i.e. it is $U(K)$.

Proof. We know that [1]

$$(\nabla_k R^j_{sks} + \nabla_s R^j_{ksk})\lambda_j = 2\,(R^j_{ksk}\varepsilon_{js} + R^j_{kss}\varepsilon_{kj}) \qquad (5)$$

When the system has two degrees of freedom and $k = 1$, $s = 2$, we have

$$R^l_{121}\varepsilon_{l2} + R^l_{122}\varepsilon_{1l} = (R^1_{121} + R^2_{122})\,\varepsilon_{12} = g^{12}\varepsilon_{12}\,(R_{1212} + R_{1212}) = 0$$

The relation (5) therefore becomes

$$(\nabla_1 R_{212j} + \nabla_2 R_{121j})\,\lambda^j = 0 \qquad (6)$$

Taking into account the fact that $R_{ijkl} = K\,(g_{ik}g_{lj} - g_{il}g_{jk})$, we find that [4]

$$\nabla_s R_{ijkl} = \nabla_s K\,(g_{ik}g_{lj} - g_{il}g_{jk})$$

which on substitution into (6) yields

$$(g_{22}g_{1l} - g_{2l}g_{12})\,\nabla_1 K\lambda^l + (g_{11}g_{2l} - g_{1l}g_{21})\,\nabla_2 K\lambda^l = 0 \qquad (7)$$

Since in the case (b) the curvature K is not constant, it follows that at least one $\nabla_s K$ is not zero. Transforming (7) we obtain

$$\nabla_1 K\lambda^1 + \nabla_2 K\lambda^2 = \nabla_s K\lambda^s = 0$$

Consequently

$$\lambda^j = \varepsilon^{js} \nabla_s K \tag{8}$$

where ε^{js} is a bivector [2]. Using the condition (2) and the relation $Q_j = \partial U / \partial q^j$, we find

$$\varepsilon^{js} \nabla_s K \nabla_j U = 0$$

which shows that U and K depend on each other, i.e. $U = U(K)$.

Conversely, let us assume that $U = U(K)$. Then for the generalized forces we find

$$Q_j = (dU / dK) \nabla_j K$$

Since the condition (8) holds, we must find whether the condition (2) also holds. Indeed,

$$\lambda^j Q_j = \varepsilon^{js} \nabla_s K \frac{dU}{dK} \nabla_j K = 0$$

which proves the theorem.

Corollary. Let, under the conditions of the Theorem, $g_{\lambda\mu}$ depend only on one of th the variables, say $g_{\lambda\mu}(q^1)$. Then [4]

$$K = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} \tag{9}$$

Both, the numerator and the denominater depend only on q^1. Consequently $K = K(q^1)$ According to the theorem $U = U(q^1)$. Thus if $g_{\lambda\mu}(q^1)$, the condition that $U(q^1)$ is necessary and sufficient for the mechanical system to have a linear integral.

Consider the example analyzed in [1]

$$2T = A\varphi^{\cdot 2} + B\theta^{\cdot 2} + 2C \cos(\theta - \varphi) \varphi^\cdot \theta^\cdot$$

$$A = {}^{16}/_3 mr^2, \qquad B = {}^4/_3 mr^2, \qquad C = mr^2$$

$$U = 3mgr \cos\varphi + mgr \cos\theta$$

Changing the variables according to the rule $q^1 = \theta - \varphi$ and $q^2 = \varphi$ we obtain

$$2T^* = A(q^{2\cdot})^2 + B(q^{2\cdot} + q^{1\cdot})^2 + 2C \cos q^1 (q^{1\cdot} + q^{2\cdot}) q^{2\cdot}$$

$$U^* = 3mgr \cos q^1 + mgr \cos(q^1 + q^2)$$

We see that $g_{\lambda\mu}(q^1)$. The force function U depends on q^2, since

$$\partial U / \partial q^2 = -mgr \sin(q^1 + q^2) \neq 0.$$

From the corollary it follows that the system has no linear integral, as was shown directly in [1].

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